On the Optimal Number of Smart Dust Particles

*Samee Ullah Khan and Muhammad Sohaib Hamid**

Department of Computer Science and Engineering, University of Texas at Arlington
Arlington, Texas 76019, USA
Email: sakhan@cse.uta.edu
*Department of Computer Engineering, National University of Science and Technology
Rawalpindi, Pakistan

***Abstract*—** *Smart Dust particles are small smart materials used for generating weather maps. We investigate the open problem posed by Vidal et al. on the optimal number of Smart Dust particles necessary for constructing precise, cost effective and accurate 3-D weather map.*

Keywords: *Smart structures, matching, optimization.*

1. INTRODUCTION

Smart Dust is small maple leaf like structure, with miniature sensors for temperature and moisture monitors and signal emitters mounted onto it. Since these leaves are very light weight, they descend slowly towards the earth's surface, and as they do they constantly send out information about temperature and moisture. Each leaf costs around $30, and is released into the atmosphere by a small auto plane [2]. Smart Dust was developed at the University of California at Berkeley under the US DARPA funding [1]. The potential application for these Smart Dust Particles as pointed out in [2,4] are to trace the wind profiles in the Bay area, and a possibility to construct 3-D weather maps. A lot of research, has been done towards the development of these particles, their functionality and their structures which is summarized in [2], there are still some problems which haunt these particles. As addressed by [4] and [3] in order to construct a 3-D weather map, we are faced with one major difficulty. The direction, from which the signal is received, is known, but not the distance, i.e. the particles, can not be mapped correctly to the 3-D map which is to be constructed from its information. In [4] authors have been able to solve this problem, by giving us an asymptotically optimal matching algorithm.

2. THE PROBLEM OF OPTIMAL MATCHING

As mentioned earlier, there exists the problem of uniquely mapping signals from various Smart Dust particles to the ground receivers. In general this matching is solvable in $O(2n \log n)$ [4]. Here, we will give another approach by transforming the problem to "Maximal-Bipartite Graph Matching Problem" (BGMP). Although we admit that our proposed approach is much slower (in solution convergence) then the approach taken up by our predecessors, but we give a solution for the generalized case on multiple receivers. Before we transform the problem to BGMP, we will quote some lemmas that are necessary for the transformation.

We can define the Bipartite Graph Matching problem as follows:

A graph $G = (V, E)$ having a set of nodes L and a set of nodes R such that:

1. $L \cap R = \phi$, $L \cup R = V$;

2. and $\forall$ $(u,v) \in E$, $u \in L$ and $v \in R$.

Lemma 1.

A matching of a graph $G = (V, E)$ is a subset of edges such that no two edges are incident to the same node.

Proof:

A matching M in a graph $G = (V, E)$ is a subset of E such that there is no $u \in V$, $v_1 \in V$, $v_2 \in V$ such that $v_1 \neq v_2$; v_1 and either $(u, v_1) \in M$ and $(u, v_2) \in M$, or $(v_1, u) \in M$ and $(v_2, u) \in M$. In other words, no node is linked to any two other nodes.

It turns out that using the approach of augmented paths, converges to a much simpler solution. Augmenting Path actually takes a non-maximum maximal matching and extends it by changing the pairing of some of the nodes. An augmenting path starts at an unmatched node, after that it alternates taking unmatched and matched edges back and forth until an unmatched node on the right is reached.

Lemma 2.

A matching in graph G is to maximum if and only if there is no augmenting path with respect to it.

Proof:

It is clear that given a Matching M, if we have an augmented path P, then an improvement can be brought to M. This can be done simply by replacing in M the edges of $P \cap M$ by $P - M$. This new M would be larger than the previous. Conversely, if M is not maximum, and there exists a larger matching M', then consider the connected components of $M \cup M'$. These are either alternating paths or alternating circuits with respect to M. At least one such component must contain more edges of M' than of M. This component is an augmenting path with respect to M.

Lemma 2 suggests the following algorithm for computing a maximum matching: start with a feasible matching M, try to find repeatedly an augmenting path P, and replace M by their Symmetric Difference Graph [5]. If there are no more paths we can be sure of the outcome and can terminate the search.

Lemma 3.

Let $G=(V,E)$ be a graph and $|V|=n$ and $|E|=m$. The bipartite matching algorithm runs in worst case time $O(nm)$ for a given graph $G=(V,E)$.

Proof:

The algorithm executes the search and augment procedures at most n times. The augment procedure clearly requires $O(n)$ time. For each node i, the search procedure performs one of the following two operations at most once: it finds an even edge, or it finds an odd one. The latter operation of course requires constant time per execution. The former operation requires $O|Adj(i)|$ (where $Adj(i)$ is the list of adjacent nodes of i), so a total of $O(\sum_{v\in V}|Adj(i))=O(m)$ time for all the nodes is needed.

Each time we augment the matching, its cardinality increases by one. If the algorithm terminates, we have a maximum matching according to Lemma 3.

First we try to find an augmenting path using a labeling technique which starts at an unmatched node p and then uses a search algorithm to identify all reachable nodes. If the algorithm finds an unmatched node, it has discovered an augmenting path. If there is no such unmatched node, there is no augmenting path starting at node p.

We will grow a search tree rooted at node p such that each path in the tree from node p to another node is an alternating path. We refer to this tree as an alternating tree and nodes in the tree are labeled nodes and the others are unlabelled. The labeled nodes are of two types: even or odd. The root node is labeled with even. Notice that whenever an unmatched node has as an odd label, the path joining the root node to this node is an augmenting path.

3. THE OPTIMAL NUMBER OF SMART DUST PARTICLES REQUIRED.

We will now address the question asked by Vidal et al. [4].

"*For a given measurement accuracy ε, what is the optimal number of leaves?*"

They conjectured that it would be: $n\approx 1/\varepsilon$. Here we will develop the mathematics for this problem, which would lead to the proof of the conjecture. Consider two different receivers R_1 and R_2 on the horizontal separated by distance d. Both receive signals from the Sender S with the same wavelength λ. The sender S is at some vertical distance making an angle θ with the vertical axis passing through the horizontal mid-point d/2.

Assume with drift and other atmospheric constraints, at any given time t, the two lengths L_1 and L_2 do not match. If so then we can find the difference of the arrival of the signals. This is known as the phase difference. We can represent the vectors in Figure 1, as complex numbers, by doing so we are able to represent the physical quantities a complex numbers. The vectors become $L_1e^{i\varphi 1}$ and $L_2e^{i\varphi 2}$ with real part as $L_1\cos\varphi_1$ and $L_2\cos\varphi_2$ by adding them we get $Le^{i\varphi L}=L_1e^{i\varphi 1}+L_2e^{i\varphi 2}$. We will now find the length of L, the complex conjugate would be the same expression as that of the normal vector addition, except that the sign's are reversed. Thus we get the following:

$$L^2=L_1^2+L_2^2+L_1L_2\left(e^{i(\varphi_1-\varphi_2)}+e^{i(\varphi_2-\varphi_1)}\right). \qquad (1)$$

Now we know that:

$$e^{i\theta}+e^{-i\theta}=\cos\theta+i\sin\theta+\cos\theta-i\sin\theta=2\cos\theta.$$

Thus, we get the final resultant as:

$$L^2=L_1^2+L_2^2+2L_1L_2\cos(\varphi_2-\varphi_1) \qquad (2)$$

The resultant ensures the effects of both the receiver's capabilities; L_1^2 would give us one of them alone, L_2^2 gives us the other one, along with the correction factor. This correction factor is the interference effect. Note that this model would also result in a *negative* correction factor. This can be rectified by rotating the receivers clockwise by π. This would make the factor positive again. But in general since the particles, would be so densely populated, there would be enough positive factors, to cancel out the effect of the negative factors.

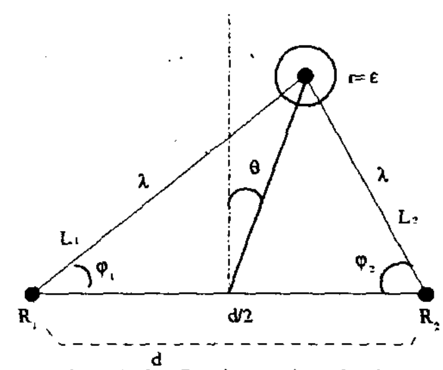

Figure 1: Two Receivers and one Sender.

We will no induce the errors, which are due to the drift and other unknown factors. Once such a scenario is reached the location of the sender S, is now within the vicinity of ε. For our convenience, we draw the wedge as a right angle triangle, although this is not necessarily the right approach, but since the error is of random nature the assumption is not of the worst possible case in any case.

By doing so we are able to compute the drift in the angle of reception for the receivers and now the vectors take the form of $L_1e^{i\left(\cosh^{(\varepsilon/L_1)}+\varphi_1\right)}$ and the angle for the receiver R_1, would be anything between:

$$\varphi_1 \text{ to } \pi - (\pi/2 + \cosh(\varepsilon / L_1)).$$

This would be similar for receiver R_2. But this is not as worst as we thought, as due to the symmetric nature of the two receivers, the effect would be cancelled out and the same result as in Equation (2). We still have not yet been able to find the exact position of the source, but that problem is answerable, by finding the phase difference (φ_1-φ_2) (the arrival of signals at receivers R_1 and R_2). We will now derive a lemma that is necessary for the proof that the phase difference solves the problem for finding the location of the sender.

Lemma 4.

If $L_1 \neq L_2$ and receivers R_1 and R_2 are d distance apart, than there would be a phase delay.

Proof:

Let there be two receivers R_1 and R_2 at distance d apart, receiving signals of the same amplitude from source S, due to the distance d and the setup angle θ over the axis at point d/2, of the legs of the outer triangle, would be larger than the adjacent leg connecting the other receiver, then there would be an intrinsic relative phase delay α, i.e. if one signal arrives at time t_0, than the other signal would arrive at time $t_0 + \alpha$.

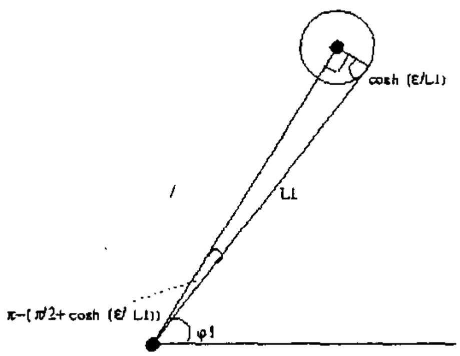

Figure 2: Receiver and Sender with drift error ϵ.

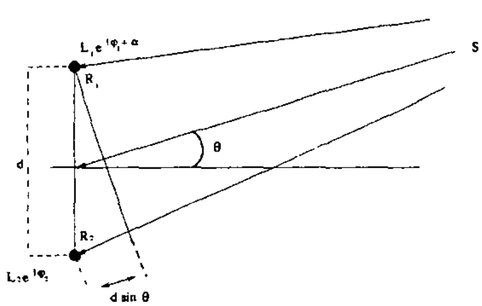

Figure 3: Receivers with phase difference α between them.

Theorem 1.

Finding the phase difference (φ_1-φ_2) gives the generalized formula of the optimal angle required for the minimized interference (maximal number of senders S_n).

Proof:

The phase relation from the lemma and Figure 3 is dsinθ, which is the difference in the distance from the source S to the two receivers R_1 and R_2, since the sender's location fluctuates with error, we thus use the factor of 2πϵ (the ball surrounding the sender in Figure 1), and thus we get:

$$\varphi = \varphi_2 - \varphi_1 = 2\pi\epsilon d \sin\theta \qquad (3)$$

In the case where the phase difference $\varphi = \pi/2$, since $\cos(\pi/2) = 0$, it follows from Equation (2) instantly that, $L^2 = L_1^2 + L_2^2$ in other words we have the exact additive distance. Equation (2) is none other than the well known Cosine law to find the third side and the special case Equation (3) the well-known Pythagorean formula for the right-angled triangle. In fact, if $\varphi \neq \pi/2$ than for the case $\varphi > \pi/2$, we have $\cos\varphi < 0$. Thus the term $2L_1L_2\cos(\varphi_2 - \varphi_1) < 0$, in Equation (2) i.e. a negative value. This is called the destructive interference. But as we had already argued that would be of little effect, as we would have a geometric symmetry and the Equation (2) would look like:

$$L^2 < L_1^2 + L_2^2. \qquad (4)$$

For the case $\varphi < \pi/2$, since $\cos\varphi > 0$, we will have $2L_1L_2\cos(\varphi_2 - \varphi_1) > 0$ and thus it would follow from Equation (2), that in the case $\varphi < \pi/2$, $L^2 > L_1^2 + L_2^2$, that is the constructive interference. Thus if we would like to have an exact measure of the angles, to which the sender should possess, we set $\varphi = \pi/2$ in Equation (3). Thus we arrive at:

$$\sin\theta_{opt} = 1/4d\epsilon \qquad (5)$$

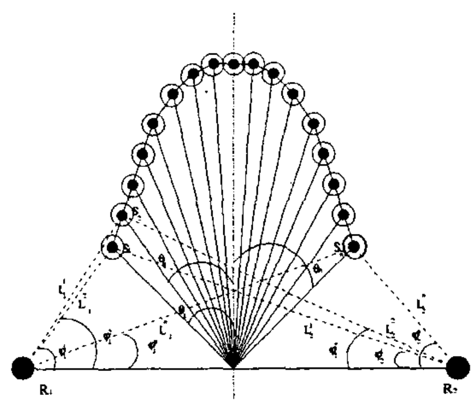

Figure 4: N Senders and 2 Receivers.

From Equation (5) we now derive some qualitative conclusions. First as $\epsilon \rightarrow 0$, notice the denominator becomes small and thus $\sin\theta_{opt}$ increases and that means $\theta\rightarrow\pi/2$. Thus lower error is optimally better along the axis connecting the receivers. If $\epsilon\rightarrow\infty$, $1/\epsilon$ becomes very small, and $\sin\theta_{opt}\rightarrow 0$ and $\theta_{opt}\rightarrow 0$. Thus higher values of ϵ are better viewed optimally perpendicular to the axis connecting the receivers. Next also intuitively true that in the values 4.d in the denominator of Equation (5), would be constant for a given set of scenario. They have virtually no effect towards the outcome of the results. Thus we can say $\sin\theta_{opt} = 1/\epsilon$ .We now just to argue that this optimized angle for one sender (which is dependent entirely on the values of ϵ), would in fact also hold for multiple senders. Thus if there are N senders, they would only be confined to a region of π, as the receivers are located on the ground. Thus the outcome of the Equation (5), would be bounded by π. The scenario is shown in Figure 4, where we spread the spectrum form N such senders, over the horizon. Each of these senders can be optimized by Equation (5). Thus we can say without doubt that indeed [4] had made the right conjecture and $n \sim 1/\epsilon$ holds.

4. CONCLUSIONS

In this paper, we addressed the problem of optimal matching and identifying the optimal number of smart dust particles needed for generating precise and cost-effective 3-D weather maps.

It would be of great interest to know if some practical data can be obtained which either confirms or negates our bounds. The bounds itself are loose and more mathematical techniques need to be applied to get a tighter bound.

REFERENCES

[1] J.M. Kahn, R.H. Katz and K.S.J. Pister. "Next century challenges: Mobile networking for smart dust", *in MOBICOM*, pages 271-278, 1999.

[2] K.S.J. Pister, J.M. Kahn and B.E. Boser. "Smart dust: wireless networks of millimeter-scale nodes", Research summary, Electronic Research Laboratory, 1999.

[3] L. Doherty. "Algorithms for position and data recovery in wireless sensor networks". Master's thesis, University of California at Berkeley, June 2000.

[4] E. Vidal, L. Longpré, V. Kreinovich and H. Haitao. "Geombinaotorics of smart dust", *Geombinatorics*, vol. 11(2), pages 54-58, 2001.

[5] N. Christofides. *Graph Theory: An Algorithmic Approach*, Academic Press, New York, USA, 1975.